Electrodynamics from an Action to the Beginning of QED

Table of contents

The unifying idea

Classical electrodynamics and quantum electrodynamics are built from the same electromagnetic variable: the four-potential A_μ. Its derivatives form the electromagnetic field tensor $F_{\mu\nu}$, and a gauge symmetry explains why different potentials can represent the same electric and magnetic fields.

The shortest route through the subject is therefore

$$\text{action} \longrightarrow A_\mu \longrightarrow F_{\mu\nu} \longrightarrow \text{gauge symmetry} \longrightarrow \text{field equations.}$$

In classical electrodynamics, the electric current is usually prescribed from outside the theory. In the classical field equations underlying QED, the source is instead a dynamical Dirac field describing charged matter. This single change produces the coupled Maxwell–Dirac equations and carries us to the point where genuinely quantum machinery must begin.

Conventions

We use natural units,

$$\hbar = c = \epsilon_0 = \mu_0 = 1,$$

and the Minkowski metric

$$\eta_{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1).$$

Greek indices run over spacetime coordinates $0, 1, 2, 3$, repeated indices are summed, and $\partial_\mu \equiv \partial/\partial x^\mu$. The four-potential and four-current are

$$A^\mu = (\phi, \mathbf{A}), \qquad J^\mu = (\rho, \mathbf{J}).$$

Other sign conventions are common. They may change the appearance of intermediate formulas but not the physics.

Part I: Classical electrodynamics

1. Begin with the action

An action assigns a number to an entire history of a physical system. The actual history is selected by the stationary-action principle, $\delta S = 0$. For the electromagnetic field coupled to a prescribed current J^μ, take

$$S[A] = \int d^4x \, \mathscr{L}, \qquad \mathscr{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - J^\mu A_\mu.$$

This compact expression contains the field dynamics and its coupling to matter. The first term is the electromagnetic field's own Lagrangian density. The second says that a current acts as a source for the potential.

At this stage, J^μ is not determined by the electromagnetic action. It might describe a known distribution of charges and currents in a wire, antenna, or beam. Treating the matter that produces this current dynamically will be the main step toward QED.

2. The four-potential and the field tensor

The electromagnetic field tensor is defined in terms of the potential by

$$\boxed{F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.}$$

It is antisymmetric: $F_{\mu\nu} = -F_{\nu\mu}$. An antisymmetric 4×4 tensor has six independent components, exactly the number contained in the three components of $\mathbf{E}$ and the three components of $\mathbf{B}$.

In three-vector language, the definition of $F_{\mu\nu}$ is equivalent to

$$\boxed{\mathbf{E} = -\nabla \phi - \frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla \times \mathbf{A}.}$$

Thus A_μ is one relativistic object that packages the scalar and vector potentials, while $F_{\mu\nu}$ packages the observable electric and magnetic fields.

3. Gauge invariance

The potential is not unique. For any sufficiently smooth scalar function $\chi(x)$, the transformation

$$\boxed{A_\mu \longrightarrow A'_\mu = A_\mu + \partial_\mu \chi}$$

leaves the field tensor unchanged:

$$F'_{\mu\nu} = \partial_\mu (A_\nu + \partial_\nu \chi) - \partial_\nu (A_\mu + \partial_\mu \chi) = F_{\mu\nu}.$$

The extra terms cancel because ordinary partial derivatives commute. Consequently, **E** and **B** are unchanged as well. A gauge transformation therefore changes the mathematical description but not the physical electromagnetic field.

The free-field term in the action is automatically gauge invariant because it depends only on $F_{\mu\nu}$. The source term changes, up to a boundary term, by

$$\Delta S_{\text{source}} = -\int d^4x\, J^\mu \partial_\mu \chi = \int d^4x\, \chi\, \partial_\mu J^\mu.$$

It is gauge invariant when the current is conserved:

$$\boxed{\partial_\mu J^\mu = 0} \qquad \Longleftrightarrow \qquad \frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0.$$

Gauge freedom is a redundancy, but a useful one. A gauge condition such as the Lorenz gauge, $\partial_\mu A^\mu = 0$, can simplify calculations without changing the measurable fields.

4. Maxwell's sourced equations from stationary action

Vary the potential by $A_\nu \to A_\nu + \delta A_\nu$, while holding the prescribed current fixed. Since

$$\delta F_{\mu\nu} = \partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu,$$

the variation of the action is, after integrating by parts and discarding a boundary term,

$$\delta S = \int d^4x\, \left(\partial_\mu F^{\mu\nu} - J^\nu\right) \delta A_\nu.$$

Stationary action for arbitrary δA_ν gives

$$\boxed{\partial_\mu F^{\mu\nu} = J^\nu.}$$

This one covariant equation contains Gauss's law and the Ampère–Maxwell law:

$$\nabla \cdot \mathbf{E} = \rho, \qquad \nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = \mathbf{J}.$$

Taking ∂_ν of the covariant equation gives $\partial_\nu J^\nu = 0$, because the contraction of the symmetric derivative $\partial_\nu \partial_\mu$ with the antisymmetric tensor $F^{\mu\nu}$ vanishes. Charge conservation is therefore required by the structure of Maxwell's equation as well as by gauge invariance of the source coupling.

5. Maxwell's homogeneous equations from the definition of $F_{\mu\nu}$

The remaining two Maxwell equations do not arise as independent Euler–Lagrange equations for A_μ. They follow identically from the way $F_{\mu\nu}$ was defined:

$$\boxed{\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0.}$$

This is the Bianchi identity. In three-vector notation it gives

$$\nabla \cdot \mathbf{B} = 0, \qquad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0.$$

The split is conceptually useful. Two Maxwell equations are equations of motion obtained from the action; the other two are identities guaranteed by expressing the fields in terms of a potential.

6. The classical structure in one view

The classical theory can now be read as a short logical chain:

1. Choose A_μ as the electromagnetic variable.
2. Build the gauge-invariant tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.
3. Form the simplest Lorentz- and gauge-invariant field term, $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$.
4. Couple the potential to a conserved external current through $-J^\mu A_\mu$.
5. Vary the action to obtain $\partial_\mu F^{\mu\nu} = J^\nu$; obtain the other Maxwell equations from the Bianchi identity.

This completes the action-based formulation of classical electromagnetism with a prescribed source.

Part II: The classical field equations underlying QED

7. Replace the prescribed current by a matter field

To make the source dynamical, introduce a Dirac spinor field $\psi(x)$. It describes charged spin-$\frac{1}{2}$ matter such as electrons at the relativistic field level. Define its Dirac adjoint by

$$\bar{\psi} \equiv \psi^\dagger \gamma^0,$$

where the gamma matrices obey

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}.$$

The free Dirac Lagrangian density is

$$\mathscr{L}_{\mathrm{D},free} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi.$$

The fields ψ and A_μ will now be varied independently. Matter produces the electromagnetic field, and the electromagnetic field acts back on matter.

8. From global phase symmetry to local gauge symmetry

The free Dirac Lagrangian is invariant under a constant phase change,

$$\psi \longrightarrow e^{-ie\chi}\psi,$$

when χ is constant. This is a global $U(1)$ symmetry. Here e is the signed electric charge carried by the field; for an electron, $e < 0$. Keeping e signed makes the formulas below independent of a separate charge-sign convention.

If χ is allowed to depend on spacetime, differentiating ψ also differentiates the phase. The ordinary derivative then fails to transform in the same way as ψ. To restore a local symmetry, introduce the covariant derivative

$$\boxed{D_\mu \equiv \partial_\mu + ieA_\mu.}$$

Under the simultaneous transformations

$$\boxed{\psi \longrightarrow \psi' = e^{-ie\chi(x)}\psi, \qquad A_\mu \longrightarrow A'_\mu = A_\mu + \partial_\mu \chi,}$$

the covariant derivative transforms like the matter field itself:

$$D'_\mu \psi' = e^{-ie\chi(x)} D_\mu \psi.$$

This is the local $U(1)$ gauge symmetry. The same gauge transformation of A_μ that appeared in classical electromagnetism is now tied to a spacetime-dependent phase transformation of charged matter.

9. The QED Lagrangian

The locally gauge-invariant Lagrangian density is

$$\boxed{\mathscr{L}_{\text{QED}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\gamma^\mu D_\mu - m)\psi.}$$

Expanding the covariant derivative makes its three pieces visible:

$$\mathscr{L}_{\text{QED}} = \underbrace{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}}_{\text{electromagnetic field}} + \underbrace{\bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi}_{\text{Dirac matter}} - \underbrace{e\bar{\psi}\gamma^\mu \psi A_\mu}_{\text{interaction}}.$$

Define the matter current

$$\boxed{j^\mu = e\bar{\psi}\gamma^\mu \psi.}$$

The interaction is then $-j^\mu A_\mu$, exactly the same form as the classical source coupling $-J^\mu A_\mu$. The difference is that j^μ is made from a field whose dynamics are included in the same action.

10. The coupled Maxwell–Dirac equations

Varying the QED action with respect to $\bar{\psi}$ gives the Dirac equation in an electromagnetic field:

$$\boxed{(i\gamma^\mu D_\mu - m)\psi = 0.}$$

Varying with respect to A_ν gives Maxwell's sourced equation:

$$\boxed{\partial_\mu F^{\mu\nu} = j^\nu = e\bar{\psi}\gamma^\nu\psi.}$$

As before, the definition of $F_{\mu\nu}$ supplies the Bianchi identity,

$$\boxed{\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0.}$$

These are the coupled Maxwell–Dirac equations. The Dirac field generates a current that sources A_μ, while A_μ, through D_μ, influences the evolution of the Dirac field. They display the mutual interaction in a closed set of classical field equations.

11. Conserved current

The global subgroup of the $U(1)$ phase symmetry leads, through Noether's theorem, to the conserved current j^μ. Using the Dirac equation and its adjoint gives

$$\boxed{\partial_\mu j^\mu = 0.}$$

The associated conserved charge is

$$Q = \int d^3x\, j^0.$$

The same conservation law is also required by the Maxwell equation: applying ∂_ν to $\partial_\mu F^{\mu\nu} = j^\nu$ again makes the left-hand side vanish identically. Gauge structure, equations of motion, and charge conservation fit together rather than appearing as separate assumptions.

12. The classical-to-QED parallel

Classical electrodynamics with an external source	Maxwell–Dirac theory underlying QED
Electromagnetic variable A_μ	The same variable A_μ
Field tensor $F_{\mu\nu}$	The same field tensor $F_{\mu\nu}$
Gauge change $A_\mu \to A_\mu + \partial_\mu \chi$	The same change, accompanied by $\psi \to e^{-ie\chi}\psi$
Prescribed conserved current J^μ	Dynamical conserved current $j^\mu = e\bar{\psi}\gamma^\mu\psi$
Coupling $-J^\mu A_\mu$	Coupling $-j^\mu A_\mu$ generated by D_μ
$\partial_\mu F^{\mu\nu} = J^\nu$	$\partial_\mu F^{\mu\nu} = j^\nu$, coupled to the Dirac equation

This parallel is the central conceptual bridge. Local phase symmetry does not merely tolerate the electromagnetic potential; it tells us how the potential must enter the matter equation through D_μ. Conversely, the resulting interaction identifies the Dirac field's conserved current as the source of the electromagnetic field.

The beginning of QED

The expression conventionally called the QED Lagrangian has now been introduced, but everything done so far can still be read as a classical theory of coupled fields. The spinor nature of ψ anticipates quantum physics, yet writing the Lagrangian and deriving the Maxwell–Dirac equations does not by itself quantize either field.

Genuine quantum electrodynamics begins when A_μ and ψ are treated as quantum fields and the theory is given a quantum interpretation. Proceeding further requires additional machinery—for example canonical quantization or path integrals, quantum states and Fock space, and usually perturbation theory. Those ideas are important, but they no longer have a simple one-to-one parallel with the classical derivation above.

This is therefore the natural stopping point for an undergraduate first pass:

$$\boxed{\text{The QED Lagrangian is the bridge; quantizing its fields is the beginning of QED.}}$$

Final summary

Both theories are organized by the same action-based language. The potential A_μ builds the gauge-invariant field strength $F_{\mu\nu}$; varying the electromagnetic action gives the sourced Maxwell equations; and the definition of $F_{\mu\nu}$ gives the homogeneous equations. In the QED Lagrangian, a local $U(1)$ phase symmetry introduces the covariant derivative and fixes the form of the electromagnetic interaction. The external classical source is replaced by the conserved current of a dynamical Dirac field, producing the coupled Maxwell–Dirac equations. Quantization is the next chapter.